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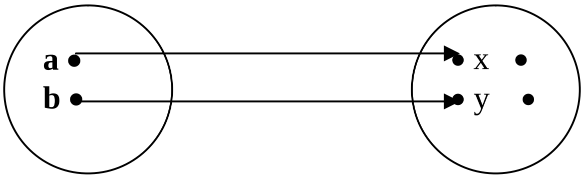
Mapping:

A mapping from a set A into a set B is a rule which assign to each element $a \in A$ a definite element $b \in B$

We say that f maps A into b and we write f: A →B.



A mapping f is also called a *function* or *Transformation* or a *Map.*

'A' is said to be the domain of f and 'B' is said to be the co-domain of f .

A map f: A →B is well-defined if

i) $a \in A \rightarrow f(a) \in B$

ii) any element $a \in A \Rightarrow$ unique element $f(a) \in B$

iii) Two or more than two elements of A may have the same image in B.

Domain & Co-domain

The domain of a mapping $f: A \rightarrow B$ is the set of element of the set A.

Co-domain of the mapping $f: A \rightarrow B$ is the set of all elements of B .

Image & Pre-image

If $a \in A$ and $b \in B$,then we say that b is the f-image of a and write $f(a) = b$. a is called the Pre-image of b.

Range or Image Set

The Range or image set is set of all f-images.

i.e. the set $\{f(x)|x \in A\}$ is denoted by $f(A)$

e.g. Let A = {1,2,3,4,5}

B = {1,2,3,4,5,6,7}

1→2, 2→3 , 3→5 , 4→ 7 , 5→7

Therefore Image set of $f(A)$ is $\{2,3,5,7\} \subset B$

Problem:

1. Let $S = \{1, 2, 3, 4\}, T = \{a, b, c, d\}$. Let us examine the following relations f_1, f_2, f_3, f_4 between S and T.

(i) $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$.

(ii) $f_2 = \{(1, a), (2, b), (3, c)\}$.

(iii) $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$.

f_1 is not a mapping from S to T since the element 1 in S is related to two different elements of T by the relation.

f_2 is not a mapping from S to T since the element 4 in S is not related to any element of T by the relation.

f_3 is a mapping from S to T. Here the image set is $\{b, c, d\}$ and it is a proper subset of the co-domain set T.

2. Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{x}\}$. Let us examine if f is a mapping from $\mathbb{R}$ to $\mathbb{R}$.

The element 0 in the domain set $\mathbb{R}$ is not related to any element of the co-domain set. Therefore f is not a mapping from $\mathbb{R}$ to $\mathbb{R}$.

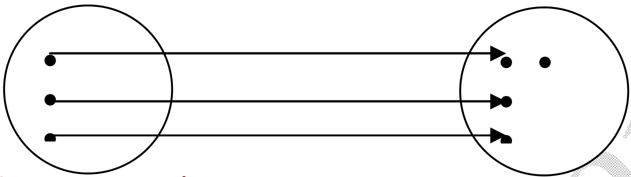
Types of Mapping

Into Mapping:

A Mapping f: A →B is said to bean into mapping if f(A) is a proper subset of B and we say that f maps A into B.

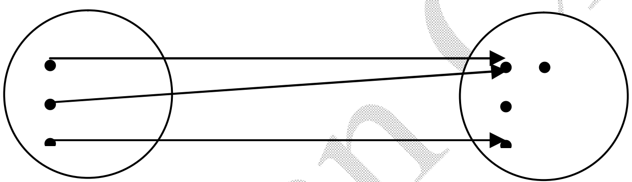
I. One-One Into Mapping

A Mapping f: A →B is said to be an One-One Into mapping if f(A) is a proper subset of B and for every f-image there is only one pre-image corresponding to that image.



II. Many-One Into Mapping

A Mapping f: A →B is said to be a Many-One Into mapping if f(A) is a proper subset of B and for at least one f-image there is two or more pre-images corresponding to that image.

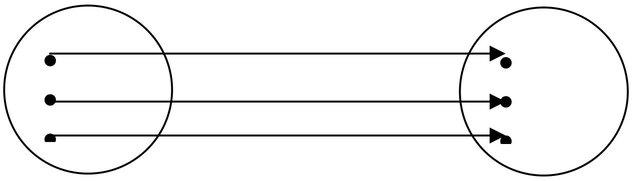


Onto Mapping:(Surjective)

A Mapping f: A →B is said to be a Onto mapping if f(A)=B and we say that f maps A onto B.

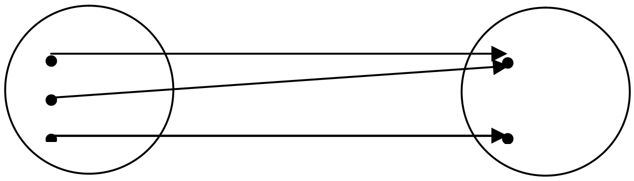
I. One-One Onto Mapping (Bijective Mapping)

A Mapping f: A →B is said to be an One-One Onto mapping if f(A) =B and for every f-image there is only one pre-image corresponding to that image.



II. Many-One Onto Mapping

A Mapping f: A →B is said to be a Many-One Onto mapping if f(A)= B and for at least one f-image there is two or more pre-images corresponding to that image.



N.B.

1. One-to-One Mapping is sometimes called Injective Mapping and this type of mapping is called Injection.

2. Onto Mapping is sometimes called Surjective Mapping.

3. One-One Onto Mapping is sometimes called Bijective Mapping and this type of mapping is called Bijection.

4. If $n(A)=p$ and $n(B)=q$ then the number of One-One Mapping from A to B is

$$= {}^{q}P_{p}\ ,\ if\ \ p\geq q$$
$$=0\ ,\ \ if\ \ p<q$$

5. If $n(A)=p$ and $n(B)=q$, where $1\leq q<p$ then the number of Onto Mapping from A to B is

$$=\sum_{r=1}^{q}(-1)^{q-r}\,{}^{q}C_{r}.r^{p}$$

Working Rule:

1. To prove One-One Mapping (Injective Mapping) : -

Step-1 : Take two arbitrary elements $x_1, x_2 \in \textit{Domain of}\ \ f$

Step- 2 : Put $f(x_1)=f(x_2)$

Step - 3 : If $f(x_1)=f(x_2)$ gives $x_1=x_2$ only then f is one-one mapping otherwise not.

Or,

Step-1 : Take two arbitrary elements $x_1, x_2 \in \textit{Domain of}\ \ f$

Step- 2 : Put $x_1 \neq x_2$

Step - 3 : If $x_1 \neq x_2$ gives $f(x_1)\neq f(x_2)$ then f is one-one mapping otherwise not.

2. To prove Onto Mapping (Surjective Mapping) or Into Mapping : -

i) If **Range** = **Co-domain** then it is onto & if **Range** ⊂ **Co-domain** then it is Into Mapping.

i.e.

i) Solve $f(x)=y$ by taking x as a function of y, say $g(y)$

ii) Now if $g(y)$ is defined & $g(y)\in \textit{domain}$ for $y\in \textit{Co-domain}$ then f is Onto mapping.

Otherwise the mapping is Into Mapping.

Worked Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 1, x \in \mathbb{R}$. Examine if f is (i) injective, (ii) surjective.

(i) Let us take two distinct elements x_1, x_2 in $\mathbb{R}$, the domain of f.
$f(x_1) = 3x_1 + 1, f(x_2) = 3x_2 + 1$.
$f(x_1) - f(x_2) = 3(x_1 - x_2) \neq 0$, since $x_1 \neq x_2$.
Since $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, f is injective.

(ii) Let us take an arbitrary element y in the set $\mathbb{R}$, the co-domain of f; and let us examine if y has a pre-image x in the domain of f.
Then $f(x) = y$ and therefore $3x + 1 = y$ or, $x = \frac{y-1}{3}$.

Since $y \in \mathbb{R}$, $\frac{y-1}{3} \in \mathbb{R}$. Therefore y has a pre-image $\frac{y-1}{3}$ in the domain of f. Since y is arbitrary, each element in the co-domain of f has a pre-image under f. Therefore f is surjective.

➢ ***Problem : If R be the set of real numbers , then discuss the mapping $f : R \to R$, where $f(x) = x^2$, $x \in R$***

Solution :

Let $x_1, x_2 \in R$, then $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply $x_1 = x_2$

$\therefore$ f is not one-one.

Let $y \in R$.

$\therefore y \in R \;\therefore y = x^2 \Rightarrow x = \pm\sqrt{y} \notin R$, for negative real number $y \in R$.

So y is not onto.

➢ ***Problem : Show that the mapping $f : R \to R$ defined by $f(x) = \dfrac{x}{x^2 - 2}$, $x \in R$ is surjective but not injective.***

Solution :

Let , $x_1, x_2 \in R$

$$\therefore f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1^2 - 2} = \frac{x_2}{x_2^2 - 2} \Rightarrow (x_1 - x_2)(x_1 x_2 + 2) = 0$$

$$\Rightarrow x_1 = x_2 \quad or \quad x_1 = -\frac{2}{x_2}$$

Therefore , we cannot say $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$\therefore$ f is not injective.

Let $y \in R$

In $y = 0$ then we get $x = 0$ in R for which $f(x) = y$

If $y \neq 0$, then we have $x = \dfrac{1 \pm \sqrt{1 + 8y^2}}{2y} \in R$ as $y \in R$ such that $f(x) = y$

$\therefore$ f is surjective.

- ***Problem : Show that the function $f : R \to R$ given by $f(x) = \cos x\ \forall\ x \in R$ is neither one-one nor onto.***

Solution :
Since $f(0) = \cos 0 = 1$ and $f(2\pi) = \cos 2\pi = 1$
Thus $f(0) = f(2\pi) = 1$
So, f is many to one function . i.e. not injective.
Since $-1 \le \cos x \le 1$, so the range of $f(x)$ is not equal to its co-domain , so f is not a surjective.

Constant Mapping

A mapping $\boldsymbol{f : A \to B}$ is called a Constant mapping (Constant Function) if $\boldsymbol{f}$ maps each element of A to a same constant element of B.

For example, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2,\ x \in \mathbb{R}$ is a constant mapping. Here $f(\mathbb{R}) = \{2\}$.

Equality Mapping

Two mapping f: A →B and g: A →C are said to be equal if f(x) = g(x) for all x∈A.
For the equality of two mappings f and g the following conditions must hold:
i) f and g have same domain D
ii) for all x∈D , f(x) = g(x).

Identity mapping

A mapping f: A →A is said to be the identity mapping on A if f(x)= x ∀ x∈A.
Let A be a non empty set and f be a mapping such that each element A is mapped on itself. Then f is called the identity mapping.
Identity mapping is always one-one onto. It is denoted by I_A.
e.g. f(x) = x
then A = {1,2,3}
f(A)= {1,2,3}
Therefore A is called identity mapping.

Inverse mapping

Let f: A →B be map and b∈B be arbitrary. Then the inverse of the element b is defined as a set consisting of these elements of A which have b as their images.
It is denoted by f^{-1}(b). Evidently, $f^{-1}(b) \subset A$
f^{-1}(b) is read as 'f inverse of b'
f^{-1}(b) = {a∈A: f(a)=b}

Note : A mapping is invertible if and only if it is One-One & Onto Mapping (Bijective Mapping).

Problem : Show that the mapping $f: R \to R$ defined by $f(x) = 3x + 5$, $x \in R$ is bijective. Determine f^{-1}.

Solution :

Let $x_1, x_2 \in R$, then $f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5 \Rightarrow x_1 = x_2$
$\therefore$ f is one-one.
Let $y \in R$.

Now $y = 3x + 5 \Rightarrow x = \dfrac{y-5}{3} \in R$ *as* $y \in R$ such that $f(x) = 3.\dfrac{y-5}{3} + 5 = y$

Therefore f is onto.
Hence f is a bijective mapping.
Therefore f has an inverse mapping $f^{-1}: R \to R$.

Since each y in the co-domain set R has a unique pre-image $\dfrac{y-5}{3}$ in the domain R.

So $f^{-1}: R \to R$ is defined by $f^{-1}(y) = \dfrac{y-5}{3}$, $y \in R$

i.e. $f^{-1}(x) = \dfrac{x-5}{3}$, $x \in R$

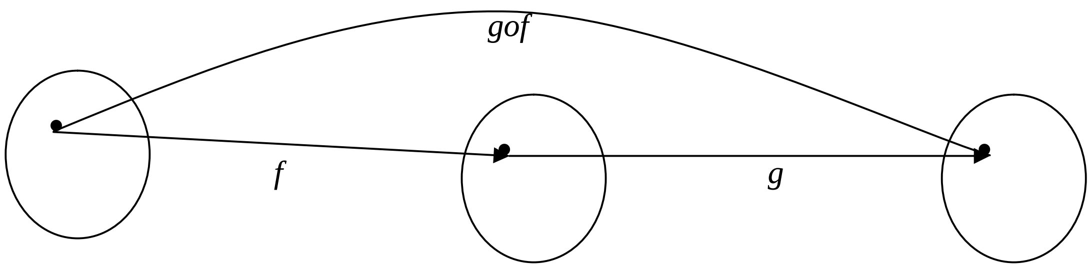
Problem: If the function f: R →R be defined by $f(x) = x^2 + 1$,then find $f^{-1}(-8)$ and $f^{-1}(17,37)$.

Solution :

$f^{-1}(-8) = \{x \in R: f(x) = -8\}$
$= \{x \in R: x^2 + 1 = -8\}$
$= \{x \in R: x^2 = -9\}$
$= \{x \in R: x = \pm 3i\}$
$= \phi$ (null set)
$f^{-1}(17,37) = \{x \in R: f(x) = 17, f(x) = 37\}$
$= \{x \in R: x^2 + 1 = 17, x^2 + 1 = 37\}$
$= \{x \in R: x^2 = 16, x^2 = 36\}$
$= \{x \in R: x = \pm 4, x = \pm 6\}$
$= \{4, -4, 6, -6\}$

Composite mapping or Product mapping

Let A,B,C be sets and let there be two mapping f and g such that f: A →B and g: B →C. Then we define the product mapping or composite mapping as *gof*: A →C such that $(gof)(x) = g[f(x)]$ $\forall$ x∈A.
It is generally denoted by *gof*.



Theorem ***Let $f : A \to B$ and $g : B \to C$ be functions***

i) If f and g are injective mapping then $g \circ f : A \to C$ is an injective.

ii) If f and g are surjective then $g \circ f$ is also surjective.

Solution :

i) $a_1, a_2 \in A$. By definition of composite mapping we have ,

$$(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$$
$$\Rightarrow f(a_1) = f(a_2)$$
$$\Rightarrow a_1 = a_2$$

Since both f and g are injectives.

Hence $g \circ f$ is injective.

ii) let $c \in C$.

Then we can find an element $a \in A$ such that $(g \circ f)(a) = c$.

Since g is onto , there is an element $b \in B$ such that $g(b) = c$.

Then , since f is onto B, there exists $a \in A$ such that $f(a) = b$.

Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$

➢ ***Problem : Let f: R →R, f(x) = sinx and g: R →R , g(x) = x^2 for all x∈R. Find fog and gof.***

Solution :

We have *gof*: R →R

(*gof*)(x) = *g{f(x)}* = *g*(sinx) = $\sin^2$x x∈R

Again, *fog* :R→R

(*fog*)(x) = *f*{*g*(x)} = *f*(x^2) = sinx^2 x∈R

∴ (*gof*)(x) ≠ (*fog*)(x)

➢ ***Problem : Let $f : X \to Y$ be an everywhere defined invertible function and A and B be arbitrary non-empty subsets of Y. Show that***

i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Proof :

i) Let $x \in X$ then

$x \in f^{-1}(A \cup B)$

$\Leftrightarrow f(x) \in A \cup B$

$\Leftrightarrow f(x) \in A \text{ or } f(x) \in B$

$\Leftrightarrow x \in f^{-1}(A) \text{ or } x \in f^{-1}(B)$

$\Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

$\therefore f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

ii) Let $x \in X$ then

$x \in f^{-1}(A \cap B)$

$\Leftrightarrow f(x) \in A \cap B$

$\Leftrightarrow f(x) \in A \text{ and } f(x) \in B$

$\Leftrightarrow x \in f^{-1}(A) \text{ and } x \in f^{-1}(B)$

$\Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$

$\therefore f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
