Relation (সম্বন্ধ)

Pre-Requisite:-

Ordered Pair:

Let A ,B be two non-empty sets and $a \in A$ *and* $b \in B$. Then (a,b) is called an ordered pair and a,b are called the first and second coordinate of the ordered pair (a,b)

Cartesian product"

The Cartesian product of the two sets A and B is denoted by A×B and it is the set of order pair of elements (a,b) where a∈A and b∈B.

i.e. A ×B = {(a,b)| a∈A and b∈B}

e.g. A = {1,2,3,4,5}

B = {5,6}

A×B = {(1,5), (1,6), (2,5), (2,6), (3,5), (3,6), (4,5), (4,6), (5,5), (5,6) }

Definition of Relation:

Let A and B be non empty sets. A relation between A and B is denoted by R (or ρ) and is a subset of A ×B.

Thus R $\subseteq A \times B$

If $(a,b) \in R$, then obviously $(a,b) \in A \times B$ and we write aRb and read as " a is related to b" .

Again if $(a,b) \notin R$ then we write a $(\sim R)$b or $a\overline{R}b$ and read as " a is not related to b"

Examples:

Let $A = \{1,2,4\}$, $B = \{4,6\}$ $\therefore A \times B = \{(1,4),(1,6),(2,4),(2,6),(4,4),(4,6)\}$

let $R = \{(1,4),(1,6),(2,4),(2,6),(4,6)\}$

Here , $R \subseteq A \times B$ and therefore R is a relation from A to B.

Domain & Range of a Relation: (ক্ষেত্র ও প্রসার)

- **Domain : (ক্ষেত্র)**

Let R be a relation from A to B . The domain of relation R is the set of all those elements $a \in A$ such that $(a,b) \in R$ form some $b \in B$.

Domain of R is denoted by Dom(R) = $\{a \in A : (a,b) \in R \text{ for some } b \in B\}$

- **Range (প্রসার)**

Let R be a relation from A to B . The Range of relation R is the set of all those elements $b \in B$ such that $(a,b) \in R$ form some $a \in A$.

Range of R = $\{b \in B : (a,b) \in R \;\; for\, some\, a \in A\}$

- ***Co-domain of a Relation (উপ ক্ষেত্র)***

If R be a relation from A to B then B is called the co-domain of relation R.

Examples

Let $A = \{1,2,3,7\}$ *and* $B = \{3,6\}$
Let aRb means a < b.
Then $R = \{(1,3),(1,6),(2,3),(2,6),(3,6)\}$
Here ,
Domain R = $\{1,2,3\}$
Range of R $\{3,6\}$,
and Co-domain of R =B = $\{3,6\}$

Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$

Example : Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$.
Then, (i) $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$
(ii) Dom (R) = $\{a, b, c\}$ = Range (R^{-1})
(iii) Range (R) = $\{1, 3\}$ = Dom (R^{-1})

Empty Relation/ Void Relation (শূন্য সম্বন্ধ)

A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$

Example:
Let $A = \{2,4,6\}, B = \{7,11\}$ and let $R = \{(a,b) : a \in A, b \in B \;\; and \;\; a - b \;\; is\, even\}$

Relation On a Set

A relation R from a non empty set A into itself is called a relation on A.
i.e. if A is a non empty set , then a subset of $A \times A$ is called a relation on A.

Examples :

Let $A\{1,2,3\}$ and Let for a $a, b \in A, aRb$ means a > b.
Then $R = \{(3,1),(3,2),(2,1)\}$ is a relation on A.

Identity Relation (একক সম্বন্ধ)

A relation R on a set A is said to be the identity relation on A if
R= $\{(a,b): a \in A, b \in A \, and \;\, a = b\}$ It is denoted by I_A.

Example:
Let $A = \{2,4,6\}$ then $I_A = \{(2,2),(4,4),(6,6)\}$

❖ *Types of RelaTion : (সম্বন্ধের প্রকারভেদ)*

Empty Relation / Void Relation (শূন্য সম্বন্ধ)

A relation R on a set A is called an empty relation or a void relation from A to A if $R = \phi$

Example:
Let $\boldsymbol{A = \{7,1,5\}}$ and let $\boldsymbol{R = \{(a,b): a \in A, b \in A \; and \; a - b \; is \; odd\}}$

Universal Relation (সার্বিক সম্বন্ধ)

A relation R on a set A is called an Universal Relation $\boldsymbol{R = A \times A}$

Reflexive Relation (স্বসম সম্বন্ধ)

Let A be a set and R be the relation defined on it . R is said to be ***Reflexive*** if $a\,R\,a$ holds for every $a \in A$.
i.e. R is ***Reflexive*** if $\langle a,a \rangle \in R$ for all $a \in A$.

Example

- "greater than or equal to" is a reflexive relation but "greater than" is not.

Symmetric Relation (প্রতিসম সম্বন্ধ)

Let A be a set and R be the relation defined on it . R is said to be ***Symmetric*** if $a\,R\,b \Rightarrow b\,R\,a$ $\forall \; (a,b) \in R$.
i.e. R is ***Symmetric*** if $\langle a,b \rangle \in R \Rightarrow \langle b,a \rangle \in R \quad \forall \; (a,b) \in R$.

Example

- "Is a blood relative of" is a symmetric relation, because *x* is a blood relative of *y* if and only if *y* is a blood relative of *x*.

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Transitive Relation (সংক্রমণ সম্বন্ধ)

Let A be a set and R be the relation defined on it . R is said to be ***Transitive*** if
$a R b$ and $b Rc \Rightarrow aRc \quad \forall a,b,c \in A$

Example

- If the straight line A is parallel to B and the straight line B is parallel to C then the straight line A is parallel to C

Antisymmetric

Let A be a set and R be the relation defined on it . R is said to be ***antisymmetric*** if
$a R b$ and $b Ra \Rightarrow a = b \quad \forall a,b \in A$

Equivalence Relation (সমতুল্যতা সম্বন্ধ)

A relation R defined in a set A is said to be an equivalence relation , iff

i) R is reflexive i.e. $aRa \; \forall \; a \in A$

ii) R is symmetric i.e. $aRb \Rightarrow bRa$

iii) R is transitive i.e. aRb and $bRc \Rightarrow aRc$

Important Results on Relation

- If R and S are two equivalence relations on a set A, then R ∩ S is also on 'equivalence relation on A.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.
- If a set A has n elements, then number of reflexive relations from A to A is 2^{n^2-n}

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Total number of relations $= 2^{n^2}$

- Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, A x B consists of mn ordered pairs. So, total number of relations from A to B is 2^{nm}.

Example

- For a parallel straight line A is parallel to the st. line A. So the relation "parallel" is reflexive.
  If the line A parallel to the line B then B is parallel to A i.e. the relation "parallel" is symmetric.
  If the straight line A is parallel to B and the straight line B is parallel to C then the straight line A is parallel to C. i.e. "parallel" is transitive.
  i.e. Here the relation "parallel" is an equivalence relation.

- Let $A = \{1,2,3\}$. Then

  i) $R_1 = \{(1,1),(2,2),(3,3),(1,3)\}$ is reflexive and transitive but not symmetric.

  ii) $R_2 = \{(1,1),(3,3),(1,3),(3,1)\}$ is symmetric and transitive but not reflexive , since $(2,2) \notin R_2$

  iii) $R_3 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$ is reflexive and symmetric but not transitive, since $(1,2) \in R_3$ and $(2,3) \in R_3$ but $(1,3) \notin R_3$

➢ ***Problem : If R be a relation in the set of integers Z defined by $R = \{(x, y) : x \in Z, y \in Z, (x - y) is\ divisible\ by\ 6\}$ then P.T. R is an equivalence relation.***

Solution:

Let $x \in Z$. Then $x - x = 0$ and 0 is divisible by 6.

Therefore , $xRx\ \forall x \in Z$

Hence , R is reflexive.

Again , $xRy \Rightarrow (x - y)$ is divisible by 6

$\Rightarrow -(x - y)$ is divisible by 6

$\Rightarrow y - x$ is divisible by 6

$\Rightarrow yRx$

Hence R is symmetric.

$xRy\ \ and\ \ yRz \Rightarrow (x - y)$ is divisible by 6 and $(y - z)$ is divisible by 6

$\Rightarrow [(x - y) + (y - z)]$ is divisible by 6

$\Rightarrow (x - z)$ is divisible by 6

$\Rightarrow xRz$

$\therefore R$ is transitive.

Thus R is an equivalence relation.

➢ ***Problem: On the set Z of integers, define a binary relation ρ by $a\rho b$ iff $a + b$ be even. Show that ρ is an equivalence relation.***

Solution :

Here the relation ρ in the set Z is defined as $\rho = \{(a,b) : a + b\, be\ even\ \forall\ a,b \in Z\}$

As, $a + a = 2a$ be even $\forall\ a \in Z$, so $a\rho a$

Therefore, ρ is reflexive.

Let $a,b \in Z$ and $a\,\rho\,b$. Then $a + b$ is even and so $b + a$ is even. Hence $b\,\rho\,a$.

$\therefore\ a\,\rho\,b \Rightarrow b\,\rho\,a\ \ \forall\, a,b \in Z \qquad \therefore\ \rho$ is symmetric.

Next , let $a,b,c \in Z$ and $a\,\rho\,b$ and $b\,\rho\,c$.

Therefore, $a + b$ and $b + c$ are even.

$\Rightarrow a + 2b + c$ is even $\Rightarrow a + c$ is even $\Rightarrow a\,\rho\,c$

$\therefore a\,\rho\,b\ \ and\ \ b\,\rho\,c \Rightarrow a\rho\,c \qquad \forall a,b,c \in Z$

$\therefore\ \rho$ is transitive.

Therefore, ρ is an equivalence relation.

- ***Problem : A relation R on the set of integers Z is defined by $R=\{(a,b): a,b\in Z \text{ and } |a-b|\le 5\}$. Is the relation reflexive , symmetric and transitive ?***

Solution : Let $a\in Z$. Then $|a-a|=0<5$. Therefore aRa holds $\forall a\in Z$. So R is reflexive.

Next, $a,b\in Z$ and aRb holds $\Rightarrow |a-b|\le 5 \Rightarrow |b-a|\le 5 \Rightarrow bRa$ holds $\forall a,b\in Z$

So, R is symmetric.

Again let $a,b,c\in Z$ and aRb *and* bRc $\Rightarrow |a-b|\le 5$ *and* $|b-c|\le 5$

Now , $|a-c|=|(a-b)+(b-c)|\le |a-b|+|b-c|\le 5+5=10$

Therefore , $|a-c|\le 5$ does not hold always.

Hence R is not transitive.s

- ***Problem : If R and S are equivalence relation on A prove that ,***

$(i) R^{-1}$ is an equivalence relation.

$(ii) R\cap S$ is an equivalence relation.

Solution :

i) Since R is an equivalence relation therefore R is reflexive , symmetric and transitive.
Let $a,b,c\in A$ be arbitrary.

The relation R^{-1} is

a) Reflexive : $(a,a)\in R^{-1}$, since aRa holds $\forall (a,a)\in R$

b) Symmetric : $(a,b)\in R^{-1}$, since $(a,b)\in R^{-1} \Rightarrow (b,a)\in R \Rightarrow (a,b)\in R$ as R is symmetric.

$\Rightarrow (b,a)\in R^{-1}$

c) Transitive : $(a,b),(b,c)\in R^{-1} \Rightarrow (a,c)\in R^{-1}$

Since , $(a,b),(b,c)\in R^{-1} \Rightarrow (b,a),(c,b)\in R$

$\Rightarrow (c,b),(b,a)\in R$

$\Rightarrow (c,a)\in R$ as R is transitive.

$\Rightarrow (a,c)\in R^{-1}$

Therefore R^{-1} is an equivalence relation.

ii) For all $a\in A, (a,a)\in R$ and $(a,a)\in S$, since R and S are equivalence relations. Hence $\forall a\in A, (a,a)\in R\cap S$. Hence $R\cap S$ is reflexive.

$(a,b)\in R\cap S \Rightarrow (a,b)\in R$ *and* $(a,b)\in S$

$\Rightarrow (b,a)\in R$ *and* $(b,a)\in S$

$\Rightarrow (b,a)\in R\cap S$

Therefore , $R\cap S$ is symmetric.

$(a,b)\in R\cap S, (b,c)\in R\cap S \Rightarrow (a,b)\in R, (b,c)\in R$ *and* $(a,b)\in S, (b,c)\in S$

$\Rightarrow (a,c)\in R$ *and* $(a,c)\in S$

$\Rightarrow (a,c)\in R\cap S$

$\therefore R\cap S$ is transitive.

Hence $R\cap S$ is an equivalence relation.

Problem : Determine the nature of the relation R on the set Z defined by aRb iff
$a, b \in Z \quad and \quad ab \geq 0$

Solution :

Let $a \in Z$. Then $a - a \geq 0$. So aRa holds $\forall a \in Z$
Therefore R is reflexive.
Next let $a, b \in Z \quad and \quad aRb$.
Then $ab \geq 0 \Rightarrow ba \geq 0 \quad \Rightarrow \quad bRa$. So R is symmetric.
Again , let $a, b, c \geq 0 \quad and \quad aRb, bRc$
Then $ab \geq 0, bc \geq 0 \Rightarrow (a.b).(b.c) \geq 0 \Rightarrow ac.b^2 \geq 0 \Rightarrow ac \geq 0$ provided $b^2 \neq 0$s
[e.g. if $a = 1, b = 0, c = -3 \ then \quad a.b = 1.0 = 0 \geq 0$; $bc = 0.(-3) = 0 \geq 0 \ but \ ac = 1.(-3) = -3 < 0$]
Hence $ab \geq 0, bc \geq 0$ do not always imply $ac \geq 0$.
So, R is not transitive.

Some more relationS

Partial Order

Let A be a set and R be the relation defined on it . R is said to be ***Partial order*** on A if it is reflexive , antisymmetric and transitive.

Linear Order

Let A be a set and R be the relation defined on it . R is said to be ***Linear order*** on A if it is
i) reflexive , antisymmetric and transitive.
ii) $aRb \ or \ bRa \quad \forall a, b \in A$

Equivalence class

For an equivalence relation ***R*** on a set ***A***, the set of the elements of ***A*** that are related to an element, say ***a***, of ***A*** is called the **equivalence class** of element ***a*** and it is denoted by **[*a*]**.

Example : For the equivalence relation of hours on a clock, equivalence classes are
[*1*] = {*1, 13, 25, ...* } = {*1+ 12n*: *n* $\in$ *N*} ,
[*2*] = {*2, 14, 26, ...* } = {*2+ 12n*: *n* $\in$ *N*} ,
....
where ***N*** is the set of natural numbers. There are altogether twelve of them.